PHILOSOPHICAL TRANSACTIONS.

I. *Researches in the Integral Calculus.—Part* II. *By* H. F. Talbot, *Esq. F.R.S.*

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§ 1.

Having explained a general method of finding the sums of integrals, I propose to apply it to discover the properties of different transcendents, beginning with those of the simplest nature.

In the first place, therefore, I will show its application to the arcs of the circle and conic sections.

As there will be frequent occasion to make use of cubic equations, I shall suppose their general form to be

$$x^3 - p\,x^2 + q\,x - r = 0.$$

When therefore the letters $p\ q\ r$ occur without explanation, it will be understood that they represent these coefficients.

§ 2. *Application to the Circle.*

Let us take the integral $\int \frac{d\,x}{1 + x^2}$, and suppose nothing to be previously known concerning the properties of the function which it represents.

Let us put, in the first place,

$$1 + x^2 = v\,x \quad \therefore x^2 - v\,x + 1 = 0.$$

The two variables $x\ y$ will be roots of this equation, so that they must satisfy the condition $x\,y = 1$. Also

$$\frac{d\,x}{1 + x^2} + \frac{d\,y}{1 + y^2} = \frac{d\,x}{v\,x} + \frac{d\,y}{v\,y} = \frac{1}{v}\,\mathrm{S}\,\frac{d\,x}{x} = 0,$$

because $\mathrm{S}\,\frac{d\,x}{x} = 0$ in any equation whose last term is constant.

$$\therefore \int \frac{d\,x}{1 + x^2} + \int \frac{d\,y}{1 + y^2} = \text{const.}$$

We thus obtain a characteristic property of the function

$$\int \frac{d\,x}{1+x^2} = f \,.\, x,$$

namely, that if $x\,y = 1$,

$$f\,x + f\,y = \text{const.}$$

The truth of which is otherwise evident, for if $x = \tan\theta$, then $y = \text{cotan}\,\theta$

$$f\,x = \theta \text{ and } f\,y = 90^\circ - \theta.$$

$$\therefore f\,x + f\,y = \text{const.}$$

Next let us investigate such a relation between *three* of these integrals that they may have an algebraic sum.

Assume $\frac{1}{1+x^2} = \frac{v+x}{a\,x}$, whence

$$x^3 + v\,x^2 + (1-a)\,x + v = 0,$$

where a is any constant quantity.

The three variables $x\,y\,z$ must be roots of this equation, which however gives only one necessary relation between them, viz.

$$x + y + z = x\,y\,z.$$

We have

$$\frac{a}{1+x^2} = \frac{v}{x} + 1$$

$$\therefore a\,\mathrm{S}\frac{d\,x}{1+x^2} = v\,\mathrm{S}\frac{d\,x}{x} + \mathrm{S}\,d\,x.$$

But $\mathrm{S}\frac{d\,x}{x} = \frac{d\,v}{v}$, and $\mathrm{S}\,d\,x = -\,d\,v$,

$$\therefore a\,\mathrm{S}\frac{d\,x}{1+x^2} = d\,v - d\,v = 0,$$

$$\therefore \mathrm{S}\int\frac{d\,x}{1+x^2} = \text{const.}$$

whence we obtain this well-known theorem in trigonometry.

If the sum of three tangents equals their product, the sum of the arcs is constant. The constant $= 180^\circ$.

Next let us suppose

$$\frac{1}{1+x^2} = \frac{v+x}{a\,x^2}$$

$$\therefore x^3 + (v-a)\,x^2 + x + v = 0.$$

This gives only one necessary relation between the roots, viz.

$$q = x\,y + x\,z + y\,z = 1.$$

For the two other coefficients $(v-a)$ and v, may be made to agree with any two arbitrary quantities. Since we have

$$\frac{a}{1+x^2} = \frac{v}{x^2} + \frac{1}{x}$$

$$\therefore a\,\mathrm{S}\frac{dx}{1+x^2} = v\,\mathrm{S}\frac{dx}{x^2} + \mathrm{S}\frac{dx}{x}.$$

But $\mathrm{S}\,\frac{1}{x} = \frac{q}{r}$, and here

$$q = 1 \text{ and } r = -v$$

$$\therefore \mathrm{S}\frac{1}{x} = -\frac{1}{v},$$

whence

$$\mathrm{S}\frac{dx}{x^2} = -\frac{dv}{v^2}$$

and

$$v\,\mathrm{S}\frac{dx}{x^2} = -\frac{dv}{v}.$$

Also

$$\mathrm{S}\frac{dx}{x} = \frac{dv}{v}.$$

Therefore

$$a\,\mathrm{S}\frac{dx}{1+x^2} = -\frac{dv}{v} + \frac{dv}{v} = 0$$

$$\therefore \mathrm{S}\int\frac{dx}{1+x^2} = \text{const.},$$

which furnishes this other well-known theorem, viz. *If three tangents are such that the sum of their products* $= 1$, *then the sum of the arcs is constant.* The constant in this case $= 90°$.

The same theorem results from the supposition

$$\frac{1}{1+x^2} = v + a\,x;$$

for this gives

$$x^3 + \frac{v}{a}x^2 + x + \frac{v-1}{a} = 0,$$

and $q = 1$ is the only necessary relation between the roots. Also

$$\mathrm{S}\frac{dx}{1+x^2} = v\,\mathrm{S}\,dx + a\,\mathrm{S}\,x\,dx.$$

But

$$\mathrm{S}\,x^2 = p^2 - 2\,q = \frac{v^2}{a^2} - 2,$$

whence

$$\mathrm{S}\,x\,dx = \frac{v\,dv}{a^2}.$$

Also

$$\mathrm{S}\,dx = -\frac{dv}{a}.$$

Therefore

$$\mathrm{S}\frac{dx}{1+x^2} = -\frac{v\,dv}{a} + \frac{v\,dv}{a} = 0. \qquad \textit{Q.E.D.}$$

These theorems, and the analogous ones which exist between any number of tangents, are well known. But when we apply the method to the integral $\int \frac{dx}{\sqrt{1-x^2}}$, we obtain relations between circular arcs which appear to be of a more novel description, and perhaps have not hitherto been noticed. Of which I will proceed to give an example. Let us suppose

$$\frac{1}{\sqrt{1-x^2}} = v\,x + 1,$$

whence

$$x^3 + \frac{2}{v}x^2 + \left(\frac{1}{v^2} - 1\right)x - \frac{2}{v} = 0.$$

In this instance the symmetrical $v = \frac{2}{r}$, and therefore making this substitution, we have

$$x^3 + r\,x^2 + \left(\frac{r^2}{4} - 1\right)x - r = 0.$$

There are therefore two necessary relations between the three roots, viz.

$$p = -r \qquad q = \frac{r^2}{4} - 1.$$

And since

$$\frac{1}{\sqrt{1-x^2}} = v\,x + 1$$

$$\mathrm{S}\,\frac{d\,x}{\sqrt{1-x^2}} = v\,\mathrm{S}\,x\,d\,x + \mathrm{S}\,d\,x.$$

But

$$\mathrm{S}\,x^2 = p^2 - 2\,q = r^2 - \left(\frac{r^2}{2} - 2\right) = \frac{r^2}{2} + 2$$

$$\therefore \mathrm{S}\,x\,d\,x = \frac{r\,d\,r}{2}$$

$$v\,\mathrm{S}\,x\,d\,x = \frac{2}{r} \cdot \frac{r\,d\,r}{2} = d\,r.$$

Also

$$\mathrm{S}\,d\,x = -\,d\,r$$

$$\therefore \mathrm{S}\,\frac{d\,x}{\sqrt{1-x^2}} = d\,r - d\,r = 0\,;$$

whence this theorem:

If the sines of three circular arcs are roots of the equation

$$x^3 + r\,x^2 + \left(\frac{r^2}{4} - 1\right)x - r = 0,$$

the sum of the arcs is constant.

I will give a numerical example of this theorem.

The value of r is arbitrary. Suppose it to be

$$= 3 - \sqrt{12} = -\,0{\cdot}4641016.$$

The roots of the equation then have the following values:

$$\begin{aligned} x &= \quad 0\cdot5 &&= \sin 30^\circ &&= \sin\theta \\ y &= \quad 0\cdot94565 &&= \sin 71^\circ\, 1' &&= \sin\theta' \\ z &= -\,0\cdot98154 &&= \sin -(78^\circ\, 59') &&= \sin\theta''\,; \end{aligned}$$

and the theorem gives the sum of the arcs, or $\mathrm{S}\,\theta =$ const. The word *sum* is used in an algebraic sense, as including the case where one or more of the arcs are to be taken *negatively*, or its definition is

$$\mathrm{S}\,\theta = \pm\,\theta \pm \theta' \pm \theta''.$$

The same ambiguity in the signs pervades the whole of this class of formulæ. In the present instance

$$\begin{aligned} \mathrm{S}\,\theta &= \theta + \theta' - \theta'' \\ &= 30^\circ + 71^\circ\, 1' + 78^\circ\, 59' = 180^\circ \end{aligned}$$

$\therefore$ the constant is a semicircle.

Ex. 2. Let $r = 0$.

$$\therefore x^3 - x = 0\,;$$

and the roots are

$$\begin{aligned} x &= 0 &&= \sin 0^\circ \\ y &= 1 &&= \sin 90^\circ \\ z &= -\,1 &&= \sin -90^\circ \end{aligned}$$

$$\therefore \theta = 0^\circ \quad \theta' = 90^\circ \quad \theta'' = -\,90^\circ\,;$$

and the same formula gives, as before,

$$\theta + \theta' - \theta'' = 180^\circ.$$

A very extensive class of formulæ respecting the arcs of the circle may be obtained in a similar manner, by applying the method more generally. Thus, if we make the supposition

$$[1.] \qquad \frac{1}{\sqrt{1 - x^2}} = a_0 + a_1\, x + \ldots\ldots + a_{n-1}\, x^{n-1},$$

where $a_0, a_1, \ldots\ldots a_{n-1}$ are constants, or any entire rational functions whatever of the variable v, we have an equation of $2\,n$ dimensions, of which x is a root.

If $x = \sin\theta_{\prime}$, and the other roots are $\sin\theta_2$, $\sin\theta_3, \ldots\ldots\ldots \sin\theta_{2\,n}$, then

$$\int \frac{d\,x}{\sqrt{1 - x^2}} = \theta_{\prime},$$

and the other integrals $= \theta_2, \theta_3 \ldots \theta_{2\,n}$. And by a direct process we obtain the final equation

$$\mathrm{S}\,\theta, \text{ or } \theta_1, + \theta_2 + \ldots\ldots + \theta_{2\,n} = f.\,v + \text{const.},$$

$f.\,v$ being an entire rational function of v.

But since it is generally admitted that no combination of circular arcs can be equal to an algebraic quantity, I conclude that we have generally

$$f.\,v = 0.$$

If we consider the generality of the supposition [1.], which admits any number of arbitrary quantities, it certainly appears remarkable that this equation $f.v = 0$ should be always verified.

§. 3. *Application to the Parabola.*

If the tangent at the vertex of a parabola be taken for the axis of abscissæ, and the semiparameter $= 1$, then if x be the abscissa, the equation of the curve will be

$$2\,y = x^2,$$

and the arc, which may be designated as arc x,

$$= \int d\,x\,\sqrt{1 + x^2}.$$

The known value of this is

$$[2.] \qquad \text{Arc}\,x = \tfrac{1}{2}\,x\sqrt{1+x^2} + \tfrac{1}{2}\log\,(x + \sqrt{1+x^2}).$$

This is a function of x, the properties of which appear to have been hitherto little examined. I will establish two theorems concerning it, which are of considerable simplicity.

Theorem I.—*If three abscissæ are the roots of the equation*

$$x^3 - r\,x^2 + \left(\frac{r^2}{4} + 1\right)x - r = 0,$$

the sum of the arcs equals the sum of the abscissæ.

Since each arc is greater than its corresponding abscissa, it is evident that the word *sum* is to be understood in an algebraic sense, or that at least one of the arcs must be taken negatively.

Theorem II.—*If three abscissæ are the roots of the equation*

$$x^3 - a\,x^2 + \left(\frac{a^2}{4} - a\,b - \frac{a^2\,b^2}{4}\right)x + \frac{a^2\,b}{2} = 0,$$

the sum of the arcs equals the product of the abscissæ.

This theorem is remarkable for its simplicity, when it is considered that it contains two arbitrary quantities, a and b, which, as it appears, may have any values.

Demonstration of Theorem I.

Put $\sqrt{1 + x^2} = x^2 + v\,x + 1$: whence

$$[1.] \qquad x^3 + 2\,v\,x^2 + (v^2 + 1)\,x + 2\,v = 0.$$

Also

$$\text{S}\,d\,x\sqrt{1+x^2} = \text{S}\,x^2\,d\,x + v\,\text{S}\,x\,d\,x + \text{S}\,d\,x$$

$$\therefore \text{S}\int d\,x\sqrt{1+x^2} = \frac{\text{S}\,x^3}{3} + \int v\,\text{S}\,x\,d\,x + \text{S}\,x.$$

The first term $\dfrac{\text{S}\,x^3}{3} = \dfrac{p^3}{3} - p\,q + r$ in all equations. Here $p = -\,2\,v \quad q = v^2 + 1 \quad r = -\,2\,v$.

$$\therefore \frac{\mathrm{S}\,x^3}{3} = \frac{-8\,v^3}{3} + 2\,v\,(v^2 + 1) - 2\,v = -\frac{2\,v^3}{3}.$$

To find the value of the second term $\int v\,\mathrm{S}\,x\,d\,x$, we have

$$\mathrm{S}\,x^2 = p^2 - 2\,q = 4\,v^2 - (2\,v^2 + 2) = 2\,v^2 - 2$$

$$\therefore \mathrm{S}\,x\,d\,x = 2\,v\,d\,v$$

and

$$\int v\,\mathrm{S}\,x\,d\,x = \int 2\,v^2\,d\,v = \frac{2\,v^3}{3}.$$

Therefore these two terms destroy each other. Consequently we have simply

$$\mathrm{S} \int d\,x\,\sqrt{1+x^2} = \mathrm{S}\,x + \mathrm{C}.$$

It appears by trial that $\mathrm{C} = 0$, and the equation between the roots [1.] becomes, by writing for v its value $\frac{-r}{2}$,

$$x^3 - r\,x^2 + \left(\frac{r^2}{4} + 1\right) x - r = 0$$

$\therefore$ the sum of three arcs $= \mathrm{S}\,x = r$. *Q.E.D.*

Example.—Let us suppose $r = 4 + 2\,\sqrt{2}$
$= 6{\cdot}828427.$

The three roots will be

$$x = 1$$
$$y = 4{\cdot}2042580$$
$$z = 1{\cdot}6241690.$$

Calculating the arcs accurately by the formula [2.], we have

$$\begin{aligned} \text{Arc } x &= 1{\cdot}147793 \\ \text{Arc } y &= 10{\cdot}156004 \\ \text{Arc } z &= 2{\cdot}179773 \end{aligned}$$

In forming the sum we must notice that arc x and arc z are to be accounted negative. Consequently we find *by subtraction,*

$$\begin{aligned} \text{Arc } y &= 10{\cdot}156004 \\ \text{Arc } x + \text{Arc } z &= 3{\cdot}327566 \\ \hline \text{Sum} &= 6{\cdot}828438 \\ r &= 6{\cdot}828427 \\ \hline \text{Error of calculation} &= 0{\cdot}000011 \end{aligned}$$

Thus the calculation verifies the theorem with considerable exactness, and shows that no constant is required to be added to the integral.

Since the sum of these three arcs is algebraic, and that each contains a logarithmic part, the sum of these three logarithms must be $= 0$: for if not, it must be an alge-

braic quantity, which is considered to be impossible. This is verified by calculation; for

$$2 \text{ arc } x = x\sqrt{1+x^2} + \log(x + \sqrt{1+x^2}).$$

Calling $\log(x + \sqrt{1+x^2}) = f.x$, we have

$$\begin{array}{rlrl} fx = & 0{\cdot}881372 & f.y = & 2{\cdot}143099 \\ fz = & 1{\cdot}261722 & fx + fz = & 2{\cdot}143094 \\ \hline & 2{\cdot}143094 & \text{sum} = & 0{\cdot}000005 \end{array}$$

This sum approaches zero very nearly. The quantities fx, fz are subtractive, being parts of 2 arc x and 2 arc z, which have been already shown to be so.

Demonstration of Theorem II.

Let $v\sqrt{1+x^2} = n\,x^2 + x + v$, where n is a constant,

$$\therefore x^3 + \frac{2}{n}x^2 + \frac{1+2\,v\,n - v^2}{n^2}x + \frac{2\,v}{n^2} = 0,$$

and

$$v\,\mathrm{S}\sqrt{1+x^2}\,.\,d\,x = n\,\mathrm{S}\,x^2\,d\,x + \mathrm{S}\,x\,d\,x,$$

the term $v\,\mathrm{S}\,d\,x$ being omitted; because, since $\mathrm{S}\,x = \frac{-2}{n}$ is constant, the factor

$$\mathrm{S}\,d\,x = 0.$$

The formula $\mathrm{S}\,x^3 = p^3 - 3\,p\,q + 3\,r$ gives

$$\frac{\mathrm{S}\,x^3}{3} = \frac{p^3}{3} - p\,q + r,$$

$\therefore$ (observing that p is constant and $= -\frac{2}{n}$) $\mathrm{S}\,x^2\,d\,x = \frac{2}{n}d\,q + d\,r$. Therefore the first term, or

$$n\,\mathrm{S}\,x^2\,d\,x = 2\,d\,q + n\,d\,r.$$

The formula $\mathrm{S}\,x^2 = p^2 - 2\,q$ gives the second term, or

$$\mathrm{S}\,x\,d\,x = -\,d\,q$$

$$\therefore n\,\mathrm{S}\,x^2\,d\,x + \mathrm{S}\,x\,d\,x = d\,q + n\,d\,r,$$

or

$$v\,\mathrm{S}\sqrt{1+x^2}\,.\,d\,x = d\,q + n\,d\,r.$$

Now we have (omitting constants),

$$q = \frac{2}{n}v - \frac{v^2}{n^2}$$

$$\therefore d\,q = \frac{2}{n}d\,v - \frac{2\,v\,d\,v}{n^2},$$

and

$$n\,d\,r = -\frac{2}{n}d\,v$$

$$\therefore d\,q + n\,d\,r = -\frac{2\,v\,d\,v}{n^2}.$$

Therefore

$$\mathrm{S}\sqrt{1+x^2}\,.\,d\,x = -\frac{2\,d\,v}{n^2}$$

$$\mathrm{S}\int\sqrt{1+x^2}\,.\,d\,x = -\frac{2\,v}{n^2} = r.$$

Now writing $n = -\frac{2}{a}$, $v = b$, we have the equation in the form given above, viz.

$$x^3 - a\,x^2 + \left(\frac{a^2}{4} - a\,b - \frac{a^2\,b^2}{4}\right)x + \frac{a^2\,b}{2} = 0\,;$$

and therefore the theorem is demonstrated.

Examples.

Ex. 1. Let $a = 2 + \sqrt{2}$, $b = 1$, the roots of the equation are

$$x = 1$$

$$y = \tfrac{1}{2}\,.\,\frac{\sqrt{5}+1}{\sqrt{2}-1}$$

$$-\,z = \tfrac{1}{2}\,.\,\frac{\sqrt{5}-1}{\sqrt{2}-1},$$

or

$$\begin{array}{rl} x = 1 & \therefore \text{ arc } x = 1{\cdot}147793 \\ y = 3{\cdot}906278 & \therefore \text{ arc } y = 8{\cdot}911399 \\ z = -\,1{\cdot}492065 & \therefore \text{ arc } z = 1{\cdot}935186 \end{array}$$

$$\therefore \mathrm{S}\,x = 3{\cdot}414213 = 2 + \sqrt{2} = a$$

and

$$x\,y\,z = -\,(3 + 2\,\sqrt{2}) = -\,5{\cdot}828426.$$

Now we have

$$\begin{array}{rl} \text{Arc } y = 8{\cdot}911399 = (1.) \\ \text{Arc } x + \text{Arc } z = 3{\cdot}082979 = (2.) \\ \hline \text{Sum } (\textit{subtractive}) = -\,5{\cdot}828420 = (2.) - (1.) \\ x\,y\,z = -\,5{\cdot}828426 \\ \hline \text{Error} = 0{\cdot}000006 \end{array}$$

The quantity which we previously called $f\,x = \log\,(x + \sqrt{1+x^2})$ has the following values:

$$\begin{array}{r} f\,x = 0{\cdot}881372 \\ f\,y = 2{\cdot}071728 \\ f\,z = 1{\cdot}190354 \\ \therefore \text{ we have } f\,x + f\,z = 2{\cdot}071726 \\ f\,y = 2{\cdot}071728 \\ \hline \text{Error} = 0{\cdot}000002 \end{array}$$

Thus it is seen that the logarithmic parts destroy each other as in the first theorem.

Ex. 2. Let $x = \frac{3}{4}$ $y = \frac{4}{3}$ be *assumed* for two roots of the equation; in which case we find $a = -\frac{5}{6}$, $b = \frac{42}{5}$, and the third root $z = -\frac{35}{12}$.

Since $xy = 1$ in this example, the theorem gives the sum of three arcs $= xyz = z$, which we propose to verify.

Now the formula

$$2 \text{ arc } x = x\sqrt{1+x^2} + \log(x + \sqrt{1+x^2})$$

gives

$$2 \text{ arc} \left(\frac{3}{4}\right) = \frac{15}{16} + \log 2$$

$$2 \text{ arc} \left(\frac{4}{3}\right) = \frac{20}{9} + \log 3$$

$$\text{the sum of which two} = \frac{455}{144} + \log 6$$

$$\text{and } 2 \text{ arc} \left(\frac{35}{12}\right) = \frac{1295}{144} + \log 6.$$

Therefore the sum (*subtractive*)

$$= -\frac{840}{144} = -\frac{70}{12}$$

$$\therefore \text{arc } x + \text{arc } y - \text{arc } z = -\frac{35}{12}$$

But on the other hand we have

$$z = -\frac{35}{12}.$$

Therefore the sum of the arcs $= z$: which was to be shown.

§ 4. *Analogous Properties of the Circle and Parabola.*

There is a manifest analogy between the *area* of the circle and the *arc* of the parabola, the former being expressed by $\int dx\sqrt{1-x^2}$, the latter by $\int dx\sqrt{1+x^2}$, which only differ in the sign. The same analogy is seen in the theorems which may be deduced respecting these integrals. Thus, for instance, the Theorem II., which we have demonstrated in the parabola, may be applied, with a slight modification, to the circle. If we put

$$v\sqrt{1 \pm x^2} = nx^2 + x + v,$$

we find the sum of three integrals of the form

$$\int dx\sqrt{1 \pm x^2} = \pm r,$$

the constant being $= 0$. The upper sign applies to the parabola, the lower to the circle. The demonstration of the latter case is omitted for brevity, being exactly similar to that of the former.

The three variables $x\,y\,z$ are roots of

$$[1.] \qquad x^3 - a\,x^2 + \left(\frac{a^2}{4} - a\,b \mp \frac{a^2\,b^2}{4}\right) x + \frac{a^2\,b}{2} = 0,$$

the upper sign applying to the parabola; a and b being two arbitrary quantities.

To exemplify this theorem in the circle.—Since

$$2\int d\,x \sqrt{1-x^2} = x\sqrt{1-x^2} + \text{arc}\sin x,$$

the theorem gives

$$-\,2\,r = \mathrm{S}\,x\sqrt{1-x^2} + \mathrm{S}\,\text{arc}\sin x.$$

The latter term, being the sum of three circular arcs, cannot form any part of the quantity $-\,2\,r$: therefore we must have this other equation,

$$\mathrm{S}\,\text{arc}\sin x = 0,$$

which we propose to verify.

Ex. 1. Suppose $a = \frac{6}{5}$, $b = \frac{2}{3}$, the equation [1.] becomes

$$x^3 - \frac{6}{5}\,x^2 - \frac{7}{25}\,x + \frac{12}{25} = 0,$$

and its roots are

$$x = \frac{4}{5}$$

$$y = 1$$

$$z = -\,\frac{3}{5}$$

$$\therefore \text{arc}\sin x = 53^\circ\ 8' = \theta$$

$$\text{arc}\sin y = 90^\circ = \theta'$$

$$\text{arc}\sin z = -\,143^\circ\ 8' = \theta''$$

$$\therefore \theta + \theta' + \theta'' = 0,$$

in accordance with the theorem.

We may assume two of the arcs arbitrarily, and thence determine the third, so as to satisfy the theorem.

Ex. 2. Thus, let $x = \frac{4}{5}$, $y = \frac{12}{13}$, we find $a = \frac{56}{65}$, $b = \frac{12}{7}$. Here it happens that $x\,y = \frac{a\,b}{2} = \frac{48}{65}$: therefore, dividing the equation

$$x\,y\,z = -\,\frac{a^2\,b}{2}$$

by the equation

$$x\,y = \frac{a\,b}{2},$$

we find the third root

$$z = -\,a.$$

Now these three values satisfy the theorem; for we have

$$\text{arc sin}\,\frac{4}{5} = 53^\circ\ 8' = \theta$$

$$\text{arc sin}\,\frac{12}{13} = 67^\circ\ 23' = \theta'$$

$$\text{arc sin} - \frac{56}{65} = -\ 120^\circ\ 31' = \theta''$$

and

$$\theta + \theta' + \theta'' = 0.$$

§ 5. *Application to the Ellipse.*

In order to obtain a relation between three elliptic integrals, the simplest supposition which we can make appears to be

$$\sqrt{\frac{1 - e^2 x^2}{1 - x^2}} = v\,x + 1,$$

whence

$$x^3 + \frac{2}{v}\,x^2 + \frac{1 - e^2 - v^2}{v^2}\,.\,x - \frac{2}{v} = 0.$$

This determines the value of the symmetrical v to be $= \frac{2}{r}$: and therefore making this substitution we have

$$x^3 + r\,x^2 + \left(\frac{1 - e^2}{4}\,.\,r^2 - 1\right) x - r = 0$$

and

$$\sqrt{\frac{1 - e^2 x^2}{1 - x^2}} = \frac{2}{r}\,x + 1,$$

whence

$$\text{S}\,d\,x\sqrt{\frac{1 - e^2 x^2}{1 - x^2}} = \frac{2}{r}\,\text{S}\,x\,d\,x + \text{S}\,d\,x.$$

But since

$$\text{S}\,x^2 = p^2 - 2\,q = r^2 - \left(\frac{1 - e^2}{2}\,.\,r^2 - 2\right) = \frac{1 + e^2}{2}\,.\,r^2 + 2$$

$$\therefore \text{S}\,x\,d\,x = \frac{1 + e^2}{2}\,r\,d\,r$$

$$\therefore \frac{2}{r}\,\text{S}\,x\,d\,x = (1 + e^2)\,d\,r.$$

Also

$$\text{S}\,d\,x = -\,d\,r$$

$$\therefore \frac{2}{r}\,\text{S}\,x\,d\,x + \text{S}\,d\,x = e^2\,d\,r$$

$$\therefore \text{S}\,d\,x\sqrt{\frac{1 - e^2 x^2}{1 - x^2}} = e^2\,d\,r$$

$$\therefore \text{S}\int d\,x\sqrt{\frac{1 - e^2 x^2}{1 - x^2}} = e^2\,r + \text{C}.$$

Or, if we suppose the radical to have a negative sign,

$$S\int d\,x\sqrt{\frac{1-e^2x^2}{1-x^2}}=C-e^2r;$$

whence the following theorem: *If three abscissæ of an ellipse are roots of the equation*

$$x^3+r\,x^2+\left(\frac{1-e^2}{4}\,.\,r^2-1\right)x-r=0,$$

the sum of the arcs $=2\,Q-e^2\,r$, *Q being the quadrant of the ellipse.*

Ex. 1. Let $e=0$, or the ellipse be a circle; the theorem then assumes this form: *If three abscissæ of a circle are roots of the equation*

$$x^3+r\,x^2+\left(\frac{r^2}{4}-1\right)x-r=0,$$

the sum of the arcs is a semicircle; the truth of which has been demonstrated previously (vide page 4.).

FAGNANI's theorem becomes illusory when $e=0$: it is therefore interesting to observe that the present theorem, on the contrary, has a real application to the circle.

Ex. 2. Let e have any value, and $r=0$; then the roots are

$$\begin{array}{ll} x=0 & \therefore\ \text{arc}\ x=0=(1.) \\ y=1 & \quad\ \text{arc}\ y=Q=(2.) \\ z=-1 & \quad\ \text{arc}\ z=-Q=(3.) \end{array}$$

and the sum, viz.

$$(1.)+(2.)-(3.)=2\,Q,$$

which is therefore the value of the constant.

Ex. 3. When x is not actually $=0$, as in the last example, but has an indefinitely small value $=\omega$, it will be found that the values of y and z differ from 1 and -1 by a quantity of the order of ω^2. But nevertheless the arcs which subtend these abscissæ differ from a quadrant of the ellipse by a quantity of the order of ω. This arises from the direction of the arc at the extremities of the axis being perpendicular to the abscissa, so that its increment is infinitely greater than that of the latter. It will be well to show the truth of the theorem in this case. When $x=\omega$ we have (putting $1-e^2=b^2$)

$$y=1-\frac{b^2\,\omega^2}{8}$$

$$z=-y.$$

For from these values we deduce $y+z=0$, and thence (neglecting quantities of the order ω^3)

$$x+y+z=x=\omega$$

$$x\,y+x\,z+y\,z=y\,z=-y^2=\frac{b^2\,\omega^2}{4}-1$$

$$x\,y\,z=-\omega.$$

So that $x\,y\,z$ are roots of

$$x^3 - \omega x^2 + \left(\frac{1-e^2}{4}\omega^2 - 1\right) x + \omega = 0,$$

which agrees with the given form by putting $r = -\omega$.

We have now to find the sum of the arcs.

The arc subtending the abscissa ($x = \omega$) may be considered as equal to it.

The arc subtending the abscissa y differs from the elliptic quadrant by an arc which may be considered equal to the *ordinate* which corresponds to y. And the same with respect to z.

Let y' be the ordinate corresponding to y. The equation of the curve gives

$$y' = b\sqrt{1-y^2};$$

but since

$$y^2 = 1 - \frac{b^2\omega^2}{4} \therefore \sqrt{1-y^2} = \frac{b\omega}{2} \quad \therefore y' = \frac{b^2\omega}{2},$$

and the arc subtending $y = Q - \frac{b^2\omega}{2}$.

The arc z has the same value. Therefore

$$\text{arc } y + \text{arc } z = 2Q - b^2\omega;$$

adding arc $x = \omega$, we have

$$\text{Sum of arcs} = 2Q + e^2\omega,$$

(or, since $\omega = -r$)

$$= 2Q - e^2 r,$$

in accordance with the theorem.

Ex. 4. Let $1 - e^2 = \frac{1}{3}$.

And also let $r = 9 - 3\sqrt{10} = -0{\cdot}4868331$;

the roots of the equation

$$x^3 + r x^2 + \left(\frac{r^2}{12} - 1\right) x - r = 0,$$

are

$$\begin{aligned} x &= 0{\cdot}5 &&= \sin 30^\circ \\ y &= 0{\cdot}98019 &&= \sin 78^\circ\, 34' \\ z &= -0{\cdot}99336 &&= \sin 83^\circ\, 24' \end{aligned}$$

Entering LEGENDRE's Table IX. with modulus $e = \sqrt{\frac{2}{3}} = \sin 54^\circ\, 44'$ and these amplitudes, we find

$$\begin{aligned} \text{arc } x &= 0{\cdot}5081 \\ \text{arc } y &= 1{\cdot}1446 \\ \text{arc } z &= 1{\cdot}1944 \\ \hline \text{Sum} &= 2{\cdot}8471 \end{aligned}$$

On the other hand we have

$$\begin{aligned} 2\,Q = 2 \text{ arc } (90^{\circ}) &= 2{\cdot}5224 \\ -\, e^2\, r &= 0{\cdot}3246 \\ \hline 2\,Q - e^2\, r &= 2{\cdot}8470 \\ \text{Sum of arcs} &= 2{\cdot}8471 \\ \hline \text{Error} &= 0{\cdot}0001 \end{aligned}$$

I will now indicate two other theorems respecting the sum of *three elliptic arcs.*

I. We may put the integral $\int d\,x \sqrt{\frac{1 - e^2\,x^2}{1 - x^2}}$ in the form

$$\int (1 + e\,x)\, d\,x \sqrt{\frac{1 - e\,x}{(1 + e\,x)\,(1 - x^2)}},$$

and assume $\frac{1 - e\,x}{(1 + e\,x)\,(1 - x^2)}$ to be a symmetrical $= \frac{1}{v}$. This gives

$$x^3 + \frac{1}{e}\, x^2 - (v + 1)\, x + \frac{v - 1}{e} = 0,$$

and the result which I find is, that if three abscissæ are the roots of this equation, the sum of the corresponding arcs $= 2\, e \sqrt{v} + C.$

II. We may put the integral in the form

$$\int \frac{d\,x}{1 + x} \sqrt{\frac{(1 + x)\,(1 - e^2\,x^2)}{1 - x}},$$

and assume $\frac{(1 + x)\,(1 - e^2\,x^2)}{1 - x} = v$, whence

$$x^3 + x^2 - \frac{v + 1}{e^2}\, x + \frac{v - 1}{e^2} = 0.$$

The result which I find is, that if three abscissæ are the roots of this equation, the sum of the arcs $= 2 \sqrt{v} + C.$

These theorems respecting the sums of elliptic arcs appear to be some of the simplest which exist; but an unlimited number of theorems of a higher order and more complicated nature are obtainable, the discussion of which would lead too far at present.

Thus if we assume

$$\sqrt{\frac{1 - e^2\,x^2}{1 - x^2}} = a_{n-1}\, x^{n-1} + a_{n-2}\, x^{n-2} + \&c.$$

where the coefficients are constants, or entire rational functions of v, we have an equation of $2\,n$ dimensions, which gives the sum of $2\,n$ elliptic arcs in terms of v.

There is no difficulty, beyond the length of the operation, in deducing these theorems, as they are all obtainable by an uniform method. But it will be of importance to show the relation between them and the previously received doctrines respecting elliptic integrals as established by LEGENDRE and others, the connexion between them not being at first sight very evident.

§ 6. *Application to the Equilateral Hyperbola.*

In order to obtain a relation between three values of the integral $\int \frac{dx}{x^3} \sqrt{1+x^4}$, which expresses the arc of the equilateral hyperbola, we may put

$$\sqrt{1+x^4} = v\,x + 1,$$

whence

$$x^3 - v^2 x - 2v = 0,$$

we have therefore $2v = r$, and making this substitution,

$$x^3 - \frac{r^2}{4} x - r = 0.$$

Also

$$\frac{\sqrt{1+x^4}}{x^2} = \frac{r}{2x} + \frac{1}{x^2}$$

$$\therefore \ \mathrm{S}\,\frac{\sqrt{1+x^4}}{x^2}\,.\,dx = \frac{r}{2}\,\mathrm{S}\frac{dx}{x} + \mathrm{S}\frac{dx}{x^2}.$$

Now we have

$$\frac{r}{2}\,\mathrm{S}\frac{dx}{x} = \frac{r}{2}\,.\,\frac{dr}{r} = \frac{dr}{2},$$

and

$$\mathrm{S}\frac{1}{x} = \frac{q}{r} = -\frac{r}{4}$$

$$\therefore \ \mathrm{S}\frac{dx}{x^2} = \frac{dr}{4}$$

$$\therefore \ \mathrm{S}\,\frac{\sqrt{1+x^4}}{x^2}\,.\,dx = \frac{dr}{2} + \frac{dr}{4} = \frac{3\,dr}{4}$$

$$\therefore \ \mathrm{S}\int \frac{\sqrt{1+x^4}}{x^2}\,.\,dx = \frac{3}{4} r + \mathrm{C};$$

so that if three abscissæ of the equilateral hyperbola are roots of the equation

$$x^3 - \frac{r^2}{4} x - r = 0,$$

the sum of the arcs $= \frac{3}{4} r + \mathrm{C}$, which is the theorem which I originally met with concerning the hyperbolic arc*.

It will be seen how very simply and directly we are conducted to it by the present method of investigation. Next let us suppose

$$-\sqrt{1+x^4} = v\,x^3 + 1,$$

whence

$$x^3 - \frac{x}{v^2} + \frac{2}{v} = 0.$$

Put $v = -\frac{2}{r}$,

* Philosophica Transactions, 1836, Part I. p. 185.

$$\therefore x^3 - \frac{r^2}{4} x - r = 0,$$

and

$$-\frac{\sqrt{1+x^4}}{x^2} = -\frac{2x}{r} + \frac{1}{x^2}.$$

Therefore

$$-\mathrm{S}\frac{\sqrt{1+x^4}}{x^2}.dx = -\frac{2}{r}\mathrm{S}\,x\,dx + \mathrm{S}\frac{dx}{x^2},$$

but

$$\mathrm{S}\frac{dx}{x^2} = \frac{dr}{4},$$

as in the last example; and

$$\mathrm{S}\,x^2 = \frac{r^2}{2} \quad \therefore -\frac{2}{r}\mathrm{S}\,x\,dx = -dr$$

$$\therefore -\mathrm{S}\frac{\sqrt{1+x^4}}{x^2}.dx = \frac{dr}{4} - dr = -\frac{3}{4}dr$$

$$\therefore \mathrm{S}\int\frac{\sqrt{1+x^4}}{x^2}.dx = \frac{3}{4}r + \mathrm{C}.$$

This result therefore agrees with the last example, and gives the same theorem, but it supplies a different demonstration of it.

We will now suppose

$$\frac{\sqrt{1+x^4}}{x^2} = 1 + \frac{a}{x} + \frac{v}{x^2},$$

a being a constant. This gives

$$x^3 + \frac{a^2+2v}{2a}x^2 + vx + \frac{v^2-1}{2a} = 0,$$

and I find this result, that if three abscissæ are roots of this equation, which may be written

$$x^3 - px^2 + qx - r = 0,$$

then the sum of the arcs

$$= p - \frac{v^2}{r} + \text{const.} = \varphi v + \mathrm{C}.$$

This sum is therefore constant if φv is so.

Let $v=k$, $v=k'$, be two values of v, which give the same value to φv, or $p - \frac{v^2}{r}$.

Let the three hyperbolic arcs in the first case be $\alpha\ \alpha'\ \alpha''$, and in the second case $\beta\ \beta'\ \beta''$, then

$$\alpha + \alpha' + \alpha'' = \beta + \beta' + \beta''.$$

All the abscissæ have the same origin at the centre of the curve, therefore the arcs have the same origin, and therefore can be subtracted from one another. Therefore putting $\alpha - \beta = \gamma$, $\alpha' - \beta' = \gamma'$, $\alpha'' - \beta'' = \gamma''$, we have

$$\gamma + \gamma' + \gamma'' = 0.$$

This appears to me to show the possibility of finding three arcs such that (neglecting

their signs) the sum of two of them shall be equal to the third (though not superposable in any part). I believe that it has been hitherto held that this equality is impossible in the ellipse and hyperbola, without the addition of *some algebraic quantity*. I should have wished therefore to have added some numerical illustration of such a result, but the length of the calculation has hitherto prevented me from doing so.